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Non–existence of equilibria with free elimination*

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Abstract This work is concerned in the existence problem of equilibria for economies with increasing returns to scale. The consequences of relaxing the free disposal assumption are investigated. It is shown that the free disposal assumption can not be dropped, not even relaxed to the assumption of free elimination, without risking that equilibria may fail to exist. This may give new insight to the work of Giraud (2000) and Jouini (1992), because they proved the existence of equilibria for economies without free disposal assuming the production sets to verify (even weaker forms of) free elimination.

Keywords Free disposal, increasing returns to scale, marginal pricing, non–convex production.

Résumé Ce travail concerne le problème de l'existence d'équilibre pour des économies aux rendements d'échelle croissants. Les conséquences d'une relaxation de l'hypothèse de la libre disposition sont étudiés. Il est montré que l'hypothèse de la libre disposition ne peut être laisser tombé, même pas relaxé á l'hypothèse de la libre élimination, sans risqué qu'il n'existe plus d' équilibres. Cela peut donner de nouvelles perspicacités aux travaux de Giraud (2000) et de Jouini (1992), car ils démontrent l'éxistence d'équilibre pour des économies sans libre disposition, supposant que les ensembles de production verifient (même des formes plus faibles) de l'hypothèse de la libre élimination.

Mot clés Libre disposition, rendements d'échelle croissants, tarification marginale, production non–convex.

Journal of Economic Literature, Classification code C500, D620.

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1 Introduction

In this paper I am concerned in production economies with increasing returns. For a survey of this subject see Brown (1991), Cornet (1988a), Quinzii (1992, 1991), and Villar (2000). The reference result concerning the existence of marginal pricing equilibria for non–convex production economies is due to Bonnisseau and Cornet (1990).

More specifically, this paper is concerned in the free disposal assumption. The economic content of the free disposal assumption is, that everything can be disposed off at no cost. The result of this paper is analogous to Koopman's (1957): it is shown that the existence of marginal (cost) pricing equilibria depends on the free disposal assumption just as Koopman (1957) showed that the existence of Walras–equilibria depends on the convexity of the production sets.

Because of the lack of economic plausibility of the free disposal assumption, economists attempt to drop or to relax the free disposal assumption. Assuming convex production sets McKenzie (1959) and Debreu (1962) proved the existence of Walras equilibria without free disposal. In the 70ties important simplifications of the proofs and generalizations in the context of convex production were published. See Bergstrom (1987) for a survey. In sum, one can say that the free disposal assumption can be dropped at no cost if production sets are assumed to be convex. However, these results are not satisfactory, because one needs to assume the production sets to be convex, which is well known to imply non increasing returns to scale. Increasing returns to scale are known to be a major sources of the *wealth of nations* since Adam Smith's (1776) famous inquiry. Therefore, in recent literature on non convex production economies an attempt was made to show the existence of equilibria without assuming free disposal. Jouini (1992) established the existence of equilibria assuming the technologies to verify *weak free elimination*. A definition of free elimination is given at assumption 2 on page 4. Notice, that Jouini (1992) even used a weaker form of the free elimination condition, then assumption 2. An example of a production set that verifies free elimination but not free disposal is displayed in figure 1 on the next page. Recently, Giraud (2000) generalized Jouini's (1992) result assuming a weaker form of free elimination, i. e. assumption 3 on page 4. Further he drops smoothness, that was assumed by Jouini (1992). A third result that relaxes the free disposal assumption is due to Hamato (1994, 1991). He supposed the production sets to be star shaped. The most general existence result as far as production is concerned is a simple consequence of a theorem due to Cornet (1988b). He showed a generalized second welfare theorem for marginal (cost) pricing equilibria, which makes no use of free disposal. In the representative consumer case this establishes the existence of marginal (cost) pricing equilibria.

In spite of these positive results, in this paper it is shown that the free disposal assumption can not simply be dropped, not even be relaxed to satisfy free elimination, without risking that equilibria may fail to exist. This fact is proven by an example in the next section. An attempt was made to construct an example that makes clear that the source of the non existence of equilibria lays the structure of the technology itself, i. e. in the vanishing Euler characteristics of the production set and the attainable set. The economy consists of one single firm. The number of consumers is two, which is minimal in view of Cornet's (1988b) result, mentioned above. The total demand is, in addition, chosen to verify the weak axiom of revealed preferences. The main reason for doing so is because since Samuelson (1948), often it was argued that the weak axiom on total demand itself (without any rationality assumption on individuals) is a better foundation of the theory of demand than the assumptions on the preferences of the consumers in the tradition of Debreu (1959). From the modern point of view the first reason for the increasing importance of the weak axiom is its empirical support. See Härtle, Hildenbrand and Jerrison (1991) at this point. Clearly, this assumption can not be derived from the standard assumptions concerning the consumers, in the tradition of Debreu (1959). Notice that one can not go any further in the following sense: If the total demand would satisfy the strong axiom of revealed preference, instead of satisfying only the weak axiom, then it is well known that the demand can be derived from a single consumer. But in this case again Cornet's (1988b) result establishes the existence of equilibria. It is assumed that there are three goods in that economy. This is the minimal number of goods possible, because

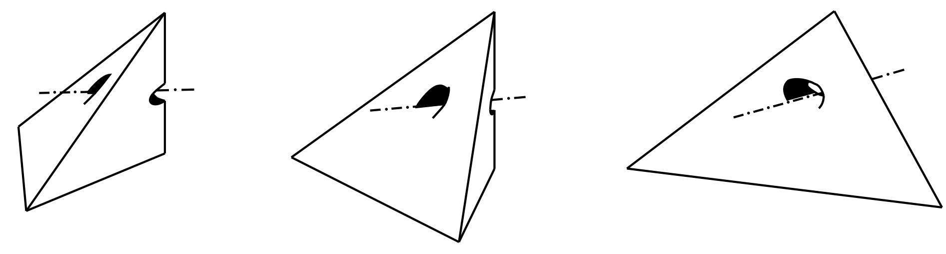

Figure 1: Attainable set of an economy that satisfies free elimination but violates free disposal

in the case of two goods the weak and the strong axiom of revealed preferences are equivalent and, as it is well known the strong axiom of revealed preferences again insures the existence of a representative consumer.

This result may give new insight to the results of Jouini (1992) and Giraud (2000). Formally all their assumptions are fulfilled but survival. However, their survival assumption guarantees the central geometrical property concerning the shape of their technologies: the Euler characteristics of the attainable production set equals unit. Note that this property ensures the existence of equilibria and can not be derived from their (other) assumptions concerning production. Of course, the Euler characteristics vanishes in the following example.

The result is linked to Kamiya's (1988) non–existence result for economies with increasing returns. The differences between both examples are: Kamiya (i) assumed the production sets to verify free disposal, (ii) relaxed the survival assumption, and (iii) did not assume the total demand to satisfy the weak axiom. His non–existence result is due to the weakness of the survival assumption. The non–existence result presented here, i. e. theorem 1 on page 8, is due to the fact that the production set does not have enough structure to imply existence in the sense of Poincare and Hopf's theorem. Namely, the Euler characteristics of the production set and of the attainable set is vanishing. There are less important differences: Kamiya gave no explicit demand. The consumption sector in this paper is completely specified. However, there is a common point in his investigation and the main result presented here: a torus. Dropping the free disposal assumption, the attainable production set of a single firm economy may be homeomorphic to a torus. Thus the set of attainable productions may have vanishing Euler characteristics. Kamiya constructed two production sets, which satisfy the free disposal assumption. The weakness of his survival assumption permitted him to construct a set of production equilibria that looks like a torus.

In the remainder of this introduction notation is introduced and Bonnisseau and Cornet's (1990) existence theorem is stated.

1.1 Notation

As far as notation is not given explicitly, the notation of Bonnisseau and Cornet (1990) and Debreu (1959) applies. The set of strictly positive numbers and the set of real numbers are denoted by $\mathbb{N}$ and $\mathbb{R}$. For $\ell \in \mathbb{N}$ and $x, y \in \mathbb{R}^\ell$ the inequalities $x > y$, $x < y$, $x \geq y$, and $x \leq y$ should be read as $x_h > y_h$, $x_h < y_h$, $x_h \geq y_h$, and $x_h \leq y_h$, for all $h \in \{1, 2, \ldots, \ell\}$. Let $\mathcal{M} \subset \mathbb{R}^\ell$ be a set, then $\partial\mathcal{M}$, $\operatorname{int}\mathcal{M}$, $\operatorname{conv}\mathcal{M}$, and $\operatorname{cone}(\mathcal{M})$ denotes the boundary, the inner, the convex hull and the smallest cone that contains the set $\mathcal{M}$. The $\operatorname{cone}(\mathcal{M})$ is defined by $\{\lambda \cdot x \mid \lambda \in \mathbb{R}_+, x \in \mathcal{M}\}$. Note that there is no reason for the $\operatorname{cone}(\mathcal{M})$ to be convex. As usual, the sets $\mathbb{R}^\ell_+$ and $\mathbb{R}^\ell_{++}$ are defined by $\{x \in \mathbb{R}^\ell \mid x \geq 0\}$ and $\operatorname{int}\mathbb{R}^\ell_+$. Definitions are indicated by the symbol $\stackrel{def}{=}$. The vector x'

denotes the transposed of the vector x and $d(x, y)$ denotes the Euclidean metric, i. e. the distance between x and y, equal to $\sqrt{(x_1 - y_1)^2 + \cdots + (x_\ell - y_\ell)^2}$. For $\mathcal{S} \subset \mathbb{R}^\ell$ the induced metric δ is defined by $\delta \stackrel{def}{=} d \mid \mathcal{S} \times \mathcal{S}$. The set $\mathcal{M} \subset \mathcal{S}$ is called relative open/relative closed (in $\mathbb{R}^\ell$), if $\mathcal{M}$ is open/closed in $\mathcal{S}$ (with respect to the induced metric δ). The boundary and the inner of the set $\mathcal{M}$ in $\mathcal{S}$ are called the relative boundary and the relative inner of $\mathcal{M}$ (in $\mathbb{R}^\ell$), denoted by $\partial_r \mathcal{M}$ and $\operatorname{int}_r \mathcal{M}$. Let P denote the unit simplex defined by $\{p \in \mathbb{R}^\ell_+ | \sum_h p_h\}$ and $q : \mathbb{R}^3_+ \to P$ the continuous function that associates at any price $p \in \mathbb{R}^\ell_{++}$, the vector in P that points in the same direction, i. e. $q(p) = (\sum_h p_h) \cdot p$. In the proof that equilibria fail to exist some trigonometric functions are used. To shorten notation the interval $[-\pi, \pi]$ is denoted by I and sin, cos : $I \to \mathbb{R}$ are abbreviated by s and c.

Some non–standard notation is necessary in this paper because more than one economy will be considered. This notation will be introduced at time needed. On the other hand, as a reference, a general notation is needed if general properties are defined. To keep the paper as readable as possible the general notation follows the classical notation quoted above. The marginal pricing rule is defined by the means of Clarke's (1983) normal cones. The normal cone in the sense of Clarke (1983) at the set Y with respect to the point $y \in Y$ is denoted by $\mathcal{N}(Y, y)$. Notice that Clarke's normal cones coincide for convex sets and for smooth manifolds with normal cones used in convex analysis and outer normal vectors used in differential geometry. The production sets considered in the results of that paper may be non–smooth, but they are always regular in the sense of Clarke (1983). See Bonnisseau and Cornet (1990), Clarke (1983, 1975) and Rockafellar (1979) for more details concerning normality and regularity in non–convex analysis. The boundary of the production set with a positive normal, i. e. the set $\{y_j \in \partial Y_j \mid \mathcal{N}(Y_j, y_j) \cap \mathbb{R}^\ell_+ \setminus 0 \neq \emptyset\}$, is denoted by $\partial^+ Y_j$. The set $\{y_j \in \partial Y_j \mid \mathcal{N}(Y_j, y_j) \cap \mathbb{R}^\ell_{++} \neq \emptyset\}$, is denoted by $\partial^{++} Y_j$.

A useful implication of the free disposal assumption is, that the normal vector to the production set at the boundary of the production set is non–negative; that is to say $\partial^+ Y_j = \partial Y_j$. This property of the free disposal assumption is used to put Bonnisseau and Cornet's (1990) assumptions in equivalent terms.

1.2 Equilibria with free disposal

In order to guarantee the existence of a marginal (cost) pricing equilibrium Bonnisseau and Cornet (1990) made the following assumptions.

Assumption 1 (free disposal) *For all firms j the production set Y_j is non–empty, closed, and satisfies free disposal: $Y_j - \mathbb{R}^\ell_+ \subset Y_j$.*

For later reference it is useful to state at this point the free elimination condition used by Jouini (1992). Recall that it is weaker then free disposal.

Assumption 2 (free elimination) *For all firms j the production set Y_j is non–empty, closed, and satisfies free elimination: $\exists y_j \in Y_j \mid y_j - \mathbb{R}^\ell_+ \subset Y_j$.*

A weaker form of the above free elimination assumption is used by Giraud (2000):

Assumption 3 (weak free elimination) *For all firms j let K_j be a compact subset of $\mathbb{R}^\ell$ and Γ_j be a convex cone with vertex zero, containing the unit vector in its interior. The set Y_j is said to verify* weak free elimination *if (i) $(\partial Y_j \setminus K_j) \cap (-\mathbb{R}^\ell_{++}) = (\partial Y_j \setminus K_j) \cap \mathbb{R}^\ell_{++} = \emptyset$ and (ii) there exists $y_j \in Y_j$ such that $y_j - \Gamma_j \subset Y_j$.*

Assumption 4 (consumers) *For all individuals i, the consumption set X_i is non–empty, closed, bounded below, and convex. The preferences are continuous, convex, transitive, and locally non–satiated.*

Assumption 5 (bounded attainables) *For all $\bar{\omega} \geq \omega$, the attainable set $\mathcal{A}(\bar{\omega})$, defined by $\{(y_j) \in \prod_{j=1}^n Y_j : \sum_{j=1}^n y_j + \bar{\omega} \in \sum_{i=1}^m X_i + \mathbb{R}^\ell_+\}$ is bounded.*

Assumption 6 (weak survival) *For all* $\bar{\omega} \geq \omega$, $\forall \big(p, (y_j)\big) \in \mathbb{R}^\ell \times \prod_{j=1}^n \partial^+ Y_j$, $p \in \bigcap_{j=1}^n \mathcal{N}(Y_j, y_j)$, $p \neq 0$ *and* $\sum_{j=1}^n y_j + \bar{\omega} \in \sum_{i=1}^m X_i + \mathbb{R}_+^\ell$ *imply* $p \cdot (\sum_{j=1}^n y_j + \bar{\omega}) > \inf p \cdot (\sum_{i=1}^m X_i + \mathbb{R}_+^\ell)$.

Assumption 7 (revenue functions) *For all individuals i (i) revenues are given by a continuous function* $r_i : \mathbb{R}_+^\ell \times \mathbb{R}^n \to \mathbb{R}$ *that satisfies* $\sum_{i=1}^m r_i\big(p, (\pi_j)\big) = p \cdot \omega + \sum_{j=1}^n \pi_j$ *(Walras law) and for all* $t > 0$*:* $r_i\big(t\,p, (t\,\pi_j)\big) = t\,r_i\big(p, (\pi_j)\big)$ *(homogeneity) and (ii) for all* $\big(p, (y_j)\big) \in \mathbb{R}_+^\ell \times \prod_{j=1}^n \partial^+ Y_j$, $p \in \bigcap_{j=1}^n \mathcal{N}(Y_j, y_j) \setminus \{0\}$ *and* $\sum_{j=1}^n y_j + \omega \in \sum_{i=1}^m X_i + \mathbb{R}_+^\ell$ *imply* $r_i\big(p, (p \cdot y_j)\big) > \inf p \cdot X_i$.

Now we can state Bonnisseau and Cornet's (1990) definition of an equilibrium:

Definition 1 *A vector* $\big(p, (x_i), (y_j)\big) \in \mathbb{R}^{\ell \cdot (n+m+1)}$ *is called a* marginal (cost) pricing equilibrium of the economy $\mathcal{E} = \big((X_i, \preceq_i, r_i), (Y_j), \omega\big)$ *if it satisfies*

(i) for all i, x_i *is a greater element for* $\preceq_i$ *in the budget set* $\big\{x \in X_i \bigm| p \cdot x \leq r_i\big(p, (p \cdot y_j)\big)\big\}$*;*

(ii) for all j, $y_j \in \partial Y_j$ *and* $p \in \mathcal{N}(Y_j, y_j) \setminus \{0\}$ *and*

(iii) $\sum_{i=1}^m x_i = \sum_{j=1}^n y_j + \omega$.

They obtain the following result:

Proposition 1 (Bonnisseau and Cornet 1990) *Suppose that the economy* $\mathcal{E} = \big((X_i, \preceq_i, r_i), (Y_j), \omega\big)$ *fulfills the assumptions 1 and 4–7, then there exists a marginal (cost) pricing equilibrium for the economy* $\mathcal{E}$*.*

Bonnisseau and Cornet's (1990) theorem is of a remarkable generality. Especially it contains Debreu's (1959) existence theorem as special case. Both results make no use of the following assumptions that are commonly made in the theory of non–convex production. They are stated here for later reference.

Assumption 8 (boundary behavior) *Let* $x : \mathbb{R}_{++}^{\ell+1} \mapsto X$ *be a correspondence and* $\{(p^\nu, w^\nu)\}_{\nu=1}^\infty$ *be a sequence in* $\mathbb{R}_{++}^{\ell+1}$*, converging to* (p, w) *such that* $w > 0$ *and* $p \notin \mathbb{R}_{++}^\ell$*. If this imply* $d\big(0, x(p^\nu, w^\nu)\big) \to \infty$*, then the correspondence x is said to verify boundary behavior.*

Recall that, if the total demand satisfies boundary behavior, i. e. assumption 8, then in equilibrium prices are strictly positive. This restricts the set of possible equilibrium production vectors to $\partial^{++} Y_j$.

In the first subsection of the introduction the role of the weak axiom was discussed. To be precise, the following applies:

Assumption 9 (weak axiom of revealed preferences) *Let* $x : \mathbb{R}_{++}^\ell \times \mathbb{R}_{++} \mapsto X$ *be the total demand correspondence and* $\bar{x}$ *and* $\hat{x}$ *be two vectors defined by* $\bar{x} = x(\bar{p}, \bar{w})$ *and* $\hat{x} = x(\hat{p}, \hat{w})$*, such that* $\bar{x} \neq \hat{x}$*. If* $\bar{p} \cdot \bar{x} \geq \bar{p} \cdot \hat{x} \Rightarrow \hat{p} \cdot \hat{x} < \hat{p} \cdot \bar{x}$*, then total demand correspondence x verifies the weak axiom of revealed preferences.*

The *weak* survival assumption, i. e. assumption 6, is very technical. Bonnisseau and Cornet developed it to encompass the existence result of Beato and Mas-Colell (1985). See remark 2.5 from Bonnisseau and Cornet (1990) at this point. Beato and Mas-Colell (1985) do not verify the following stronger, more common, and perhaps also more intuitive survival assumption:

Assumption 10 (survival assumption) *For every* $\big(p, (y_j)\big) \in \mathbb{R}_+^\ell \times \prod_j \partial^+ Y_j$, $p \in \cap_j \mathcal{N}(Y_j, y_j) \setminus \{0\}$ *implies* $p \cdot (y + \omega) > \inf p \cdot (\sum_i X_i + \mathbb{R}_+^\ell)$.

The production sets may be non–smooth. However, they may verify the following

Assumption 11 (regular production) *For all firms j the boundary* ∂Y_j *is regular.*

2 Non–existence of equilibria with free elimination

The existence results without free disposal may suggest that the free disposal assumption can be relaxed to free elimination or even be neglected at no cost. The aim of this paper is to present an example that shows that this conclusion is misleading.

2.1 The economies

This subsection starts with a rather technical description of the demand side of the economies considered in this paper. It follows a description of the technologies and a definition of the economies. The subsection ends stating the main result and a sketch of the proof.

The demand, considered in that paper, is based on Shafer's (1974) well known example of a demand function satisfying the weak axiom of revealed preferences but not the strong:

$$\begin{aligned} x_h &= \frac{w}{2\,p_h\left(1+\sqrt{\frac{p_2}{p_1}}\right)} \qquad h=1,2 \\ x_3 &= \frac{w}{p_3\left(1+\sqrt{\frac{p_1}{p_2}}\right)}. \end{aligned} \tag{1}$$

Recall that Shafer (1974) did not derive this demand function from individual preferences and income, as it is stated at assumptions 4 and 7. Therefore, at lemma 1, two preference relations $\preceq_i$ and revenue functions r_i are defined, that aggregate to Shafer's demand. However, there remains a technical problem: with these revenues, the income of the first/second individual vanishes, if the price of first/second commodity vanishes, even if total income is strictly positive and prices and productions are in production equilibrium, i. e. $p \in \mathcal{N}(Y, y)$. This is a violation of assumed fairness of the distribution of incomes, i. e. assertion (ii) of assumption 7 on page 5. To overcome this problem, revenues are changed if the first two normalized prices are smaller then some ϵ. For that reason parameterized revenue and demand functions (depending on ϵ) are defined at lemma 1. It will be seen later, that it is possible to choose $\epsilon > 0$ small enough such that no equilibria were added by this change.

Lemma 1 *Let the preference relations $\preceq_i$, the revenue functions $r_i : \mathbb{R}^3_+ \times \mathbb{R} \to \mathbb{R}$, and the parameterized revenue functions $r^\epsilon_i : [0, 1/\sqrt{3}] \times \mathbb{R}^3_+ \times \mathbb{R} \to \mathbb{R}$ be defined by*

$$\begin{aligned} (x_{1,1}, x_{1,2}, x_{1,3}) \preceq_1 (\hat{x}_{1,1}, \hat{x}_{1,2}, \hat{x}_{1,3}) \quad &\textit{iff} \quad x_{1,3} \le \hat{x}_{1,3}, \\ (x_{2,1}, x_{2,2}, x_{2,3}) \preceq_2 (\hat{x}_{2,1}, \hat{x}_{2,2}, \hat{x}_{2,3}) \quad &\textit{iff} \quad x_{2,1}\cdot x_{2,2} \le \hat{x}_{2,1}\cdot \hat{x}_{2,2}, \end{aligned}$$

$$r_1 \stackrel{def}{=} \frac{w}{1+\sqrt{\frac{p_1}{p_2}}} \qquad \textit{and} \qquad r_2 \stackrel{def}{=} \frac{w}{1+\sqrt{\frac{p_2}{p_1}}}, \tag{2}$$

$$r^\epsilon_1 \stackrel{def}{=} \begin{cases} \frac{w}{1+\sqrt{p_1/p_2}} & \textit{if} \quad q_1 \ge \epsilon, q_2 \ge \epsilon \\ \frac{w}{1+\sqrt{q_2/\epsilon}} & \textit{if} \quad q_1 \ge \epsilon, q_2 < \epsilon \\ \frac{w}{1+\sqrt{\epsilon/q_2}} & \textit{if} \quad q_1 < \epsilon, q_2 \ge \epsilon \\ \frac{w}{2} & \textit{if} \quad q_1 < \epsilon, q_2 < \epsilon \end{cases} \qquad r^\epsilon_2 \stackrel{def}{=} \begin{cases} \frac{w}{1+\sqrt{p_2/p_1}} & \textit{if} \quad q_1 \ge \epsilon, q_2 \ge \epsilon \\ \frac{w}{1+\sqrt{\epsilon/q_1}} & \textit{if} \quad q_1 \ge \epsilon, q_2 < \epsilon \\ \frac{w}{1+\sqrt{q_2/\epsilon}} & \textit{if} \quad q_1 < \epsilon, q_2 \ge \epsilon \\ \frac{w}{2} & \textit{if} \quad q_1 < \epsilon, q_2 < \epsilon \end{cases} \tag{3}$$

Then (a): assumption 4 on page 4 is verified for $\epsilon > 0$ and (b): the individual demand x^ϵ_i that are greater elements for $\preceq_i$ in the budget set $\left\{x \in X_i | p\cdot x \le r^\epsilon_i\big(\epsilon, p, (p\cdot y_j)\big)\right\}$, i. e. that satisfies condition (i) of definition 1 on the page before, are given by the functions $x^\epsilon_i : [0, 1/\sqrt{3}] \times \mathbb{R}^3_{++} \times \mathbb{R}_{++} \to \mathbb{R}^3_{++}$ equal to

$$x^\epsilon_1 = \left(0, 0, \frac{r^\epsilon_1}{p_3}\right) \quad \textit{and} \quad x^\epsilon_2 = \left(\frac{r^\epsilon_2}{2\,p_1}, \frac{r^\epsilon_2}{2\,p_2}, 0\right), \tag{4}$$

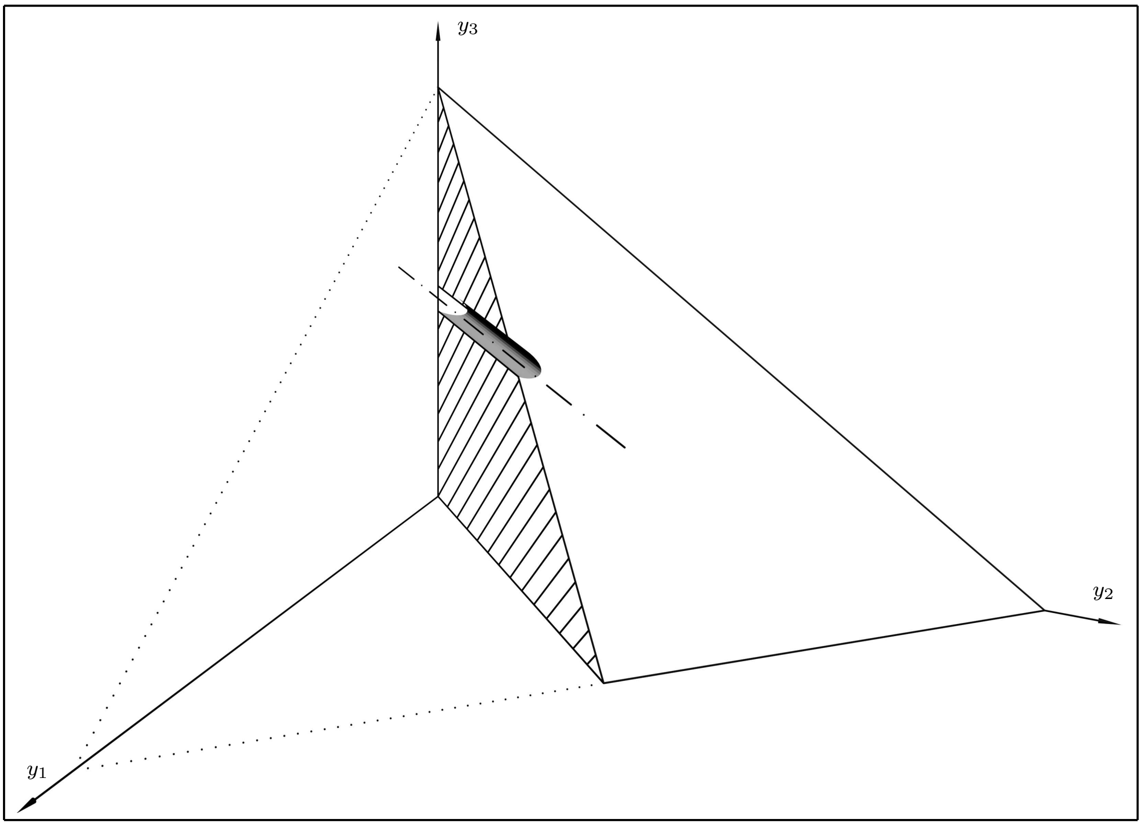

Figure 2: The attainable set, cutted

(c) In case $\epsilon = 0$ follows $x^\epsilon|_{\epsilon=0} = x$.

Proof (a) The set $X_i = \mathbb{R}^3_+$ is well known to be non–empty, closed, bounded below and convex. Preferences of the first and the second consumer are linear in the third commodity respectively of Cobb–Douglas type in both first two commodities. These preferences are well known to be continuous, convex, transitive and locally non–satiated. Moreover, preferences are monotonic. (b) These preferences of the consumers yield the parameterized demand functions claimed at equation (4). (c) Notice that $\epsilon = 0$ implies $r_i^\epsilon = r_i$. Thus $x^\epsilon|_{\epsilon=0} = x$. This proves the lemma. Q. E. D.

In this paper initial endowments are thought to be used for production only. Therefore

$$\omega \stackrel{def}{=} 0 \tag{5}$$

is assumed. Let the vectors p^{lin}, y^{lin} and the function $g_1 : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$p^{lin} \stackrel{def}{=} (1, 1, \sqrt{2}) \quad \text{and} \quad y^{lin} \stackrel{def}{=} (2, 2, 2 \cdot \sqrt{2}) \tag{6}$$

$$g_1(y) \stackrel{def}{=} y \cdot p^{lin} - 8. \tag{7}$$

Then the set Y^{lin}, i. e. the convex set that contains y^{lin} in the boundary and admits only p^{lin} as a normal, can be defined by

$$Y^{lin} \stackrel{def}{=} \{y \in \mathbb{R}^3 \mid g_1(y) \leq 0\}. \tag{8}$$

Let the set D_a be defined by $D_a \stackrel{def}{=} \{y \in \mathbb{R}^3 \mid y_1 - y_2 \in [-1,1], y_3 \in [3/\sqrt{2}, 5/\sqrt{2}]\}$, and the functions $a : D_a \to \mathbb{R}$ and $b : \mathbb{R}^3 \to \mathbb{R}$

$$a \stackrel{def}{=} 10 \left(\sqrt{\frac{\sqrt{2}y_3 - 8 - (y_1 - y_2)}{\sqrt{2}y_3 - 8 + (y_1 - y_2)}} - 1 \right) \qquad \text{and} \qquad b \stackrel{def}{=} 5\,(2\sqrt{2} - y_3) \tag{9}$$

Further, $D \stackrel{def}{=} \{y \in D_a \mid a^2 + a\,b + b^2 \leq 1\}$ and let the function $g_2 : \mathbb{R}^3 \mapsto \mathbb{R}$ and the sets $\mathcal{T}$, $\hat{Y}$, and Y^0 be defined by

$$g_2(y) \stackrel{def}{=} \begin{cases} \frac{3}{4} - a^2 - a\,b - b^2 & \text{if } y \in D, \\ -\frac{1}{4} & \text{else,} \end{cases} \tag{10}$$

$$\mathcal{T} \stackrel{def}{=} \{y \in \mathbb{R}^3 \mid g_2(y) \leq 0\}, \tag{11}$$

$$\hat{Y} \stackrel{def}{=} Y^{lin} \cap \mathcal{T} = \{y \in \mathbb{R}^3 \mid g_1(y) \leq 0, g_2(y) \leq 0\}, \qquad \text{and} \tag{12}$$

$$Y^0 \stackrel{def}{=} \partial Y^{lin} \cap \partial\mathcal{T} = \{y \in \mathbb{R}^3 \mid g_1(y) = 0, g_2(y) = 0\}. \tag{13}$$

It may not be obvious, that the functions a and g_2 are well defined. This fact will be proven at lemma 5 on page 12. A glimpse at figure 2 on the page before may give you a geometrical intuition of the production set $\hat{Y}$. To better view the boundary of the tube $\mathcal{T}$, the attainable set is cutted along the hyper-plan $\{y \in \mathbb{R}^3 \mid y_1 = y_2\}$. For orientation, the edges of the attainable set of the linear economy $\mathcal{E}^{lin}$ are represented by dotted lines. The slash dotted line is the axis $\{y \in \mathbb{R}^3 \mid y_1 = y_2, y_3 = 2 \cdot \sqrt{2}\}$. It follows from lemma 4, that this line passes through the equilibrium supply of the linear economy $\mathcal{E}^{lin}$.

Now, the main result of this paper can be formulated:

Theorem 1 *There exists an $\epsilon > 0$ such that the economy $\hat{\mathcal{E}}^\epsilon \stackrel{def}{=} \big((\mathbb{R}^3_+, \preceq_i, r_i^\epsilon, \epsilon), \hat{Y}, \omega\big)$ fulfills the assumptions 2–11 but does not admit marginal (cost) pricing equilibria.*

Sketch of the proof This theorem is proven in lemma size. The lemmas are grouped in three subsections. Each subsection corresponds to one step of the proof. The first step, lemma 1–7, is to derive that the economies $\hat{\mathcal{E}}^\epsilon$ match the assumptions 2–11. Afterwards, at lemma 8 and 9, the problem is simplified: it is shown to be sufficient to establish the non existence of equilibria simpler economy $\mathcal{E}^0$, that will be defined later. There are two main differences between these economies. (1) The production set is smaller, namely Y^0 instead of $\hat{Y}$ and (2) the demand is simpler, namely Shafer's (1974) demand as stated at equation (1) instead of x^ϵ as stated at equation (4). These simplifications allow at lemma 10 to establish that the economy $\mathcal{E}^0$ does not admit an equilibrium. This proves the theorem.

2.2 Properties of the economies $\hat{\mathcal{E}}^\epsilon$

In this subsection it is proven that the economies $\hat{\mathcal{E}}^\epsilon$ verify the assumptions of theorem 1, i. e. match the assumptions 2–11. The ordering of the lemmas is prescribed, to some extend, by technical necessities: to avoid circular arguments the ordering of the results is organized such that no result is used, until it is proven. It may not be easy to keep track of the completeness of the result. To simplify that task the following table indicates which assumption is proven at which lemma:

Assumption	2	4	5	7	8	9	10	11
Lemma	5	1	6	1	2	2	7	3

Notice that assumption 3 and 6 do not need to be verified because they follow from assumption 2 and 10. Weak free elimination, i. e. assumption 3, was developed by Giraud (2000) to generalize Jouini's (1992) existence result. As already mentioned, Bonnisseau and Cornet's (1990) weak survival assumption, i. e. assumption 6, is weaker then assumption 10.

Lemma 2 *Let the consumers be defined as in lemma 1 on page 6. Then, for* $\epsilon \in \left]0, 1/\sqrt{3}\right[$, *the normalized total demand verifies (1) boundary behavior and (2) the weak axiom of revealed preferences, i. e. assumptions 8 and 9.*

Proof (1) Let $\epsilon \in]0, 1/\sqrt{3}[$ be given and let $\{p^\nu, w^\nu\}_{\nu=1}^{\infty}$ be a sequence with strictly positive elements. The sequence is supposed to converge. The limit is denoted by (p, w). It admits the properties $w > 0$, $p \notin \mathbb{R}_{++}$ and $p \neq 0$. Further, let r_i^ν be defined by $r_i^\nu \stackrel{def}{=} r_i^\epsilon(\epsilon^\nu, p^\nu, w^\nu)$. The function r_i^ϵ is continuous. Thus the sequences $\{r_i^\nu\}_{\nu=1}^{\infty}$ converge towards some points r_i. From the assumption $w > 0$ follows, by the definition of r_i at equation (3) on page 6 that $r_i > 0$. In this case the definition of x_i^ϵ at equation (4) on page 6 guarantees that $p_h \to 0$ implies $x_h^\epsilon = x_{1,h}^\epsilon + x_{2,h}^\epsilon \to \infty$. Thus the total demand satisfies boundary behavior.

(2) Due to the homogeneity of the total demand, one can assume, without loss of generality, that total income equals unit. Denote $\bar{q} = q(\bar{p})$, $\hat{q} = q(\hat{p})$ $\bar{r}_i = r_i^\epsilon(\bar{p}, 1)$, $\hat{r}_i = r_i^\epsilon(\hat{p}, 1)$, $\bar{x} = \left(\bar{r}_2/(2\,\bar{p}_1), \bar{r}_2/(2\,\bar{p}_2), \bar{r}_1/\bar{p}_3\right)$, and $\hat{x} = \left(\hat{r}_2/(2\,\hat{p}_1), \hat{r}_2/(2\,\hat{p}_2), \hat{r}_1/\hat{p}_3\right)$, with r_i^ϵ defined at equation (3) on page 6. Compare $\bar{x}$ and $\hat{x}$ to the definition of x_i^ϵ at equation (4) on page 6 to see that it yields total demand for prices $\bar{p}, \hat{p} \in \mathbb{R}^3_{++}$. The main part of the proof is to establish the inequality (i): $(\hat{p} \cdot \bar{x})(\bar{p} \cdot \hat{x}) \geq 1$. From inequality (i) one can derive that the total demand x^ϵ verifies the weak axiom by simply showing that (i) is a strict inequality, i. e. the case $(\hat{p} \cdot \bar{x})(\bar{p} \cdot \hat{x}) = 1$ does not occur.

One admits readily that the sum of a real positive value and its inverse is greater or equal to two. Thus, for all $\bar{p}, \hat{p} > 0$, the inequality $\left((\hat{p}_1\,\bar{p}_2)/(\bar{p}_1\,\hat{p}_2) + 2 + (\bar{p}_1\,\hat{p}_2)/(\hat{p}_1\,\bar{p}_2)\right)/4 \geq 1$ holds. This implies

$$\left(\frac{\bar{p}_3}{\hat{p}_3}\left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2}\right)\right)\left(\frac{\hat{p}_3}{\bar{p}_3}\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)\right) \geq 1. \tag{14}$$

The assumption $\bar{p}, \hat{p}, \bar{w}$, and $\hat{w} > 0$ implies $\bar{r}_1, \hat{r}_1 \in]0, 1[$. From the fact

$$\begin{aligned}&\left(\frac{\bar{p}_3}{\hat{p}_3}\left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2}\right)\right)^{\bar{r}_1}\left(\frac{\hat{p}_3}{\bar{p}_3}\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)\right)^{\hat{r}_1} \in \\ &\quad\left[\left(\frac{\bar{p}_3}{\hat{p}_3}\left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2}\right)\frac{\hat{p}_3}{\bar{p}_3}\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)\right)^{\bar{r}_1}, \left(\frac{\bar{p}_3}{\hat{p}_3}\left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2}\right)\frac{\hat{p}_3}{\bar{p}_3}\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)\right)^{\hat{r}_1}\right]\end{aligned} \tag{15}$$

follows, by the mean value theorem, that there exists some $\mu \in [\hat{r}_1, \bar{r}_1] \subset]0, 1[$ such that

$$\begin{aligned}&\left(\frac{\bar{p}_3}{\hat{p}_3}\left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2}\right)\right)^{\bar{r}_1}\left(\frac{\hat{p}_3}{\bar{p}_3}\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)\right)^{\hat{r}_1} = \\ &\qquad\qquad\left(\frac{\bar{p}_3}{\hat{p}_3}\left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2}\right)\frac{\hat{p}_3}{\bar{p}_3}\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)\right)^{\mu}\end{aligned} \tag{16}$$

The fact $\mu \in]0, 1[$ and equation (14) together yields

$$\left(\frac{\bar{p}_3}{\hat{p}_3}\left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2}\right)\right)^{\bar{r}_1}\left(\frac{\hat{p}_3}{\bar{p}_3}\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)\right)^{\hat{r}_1} \geq 1.$$

Using the first time Walras' law, i. e. the fact $r_1 + r_2 = 1$, the above inequality is equivalent to

$$\frac{\hat{p}_3{}^{\bar{r}_2}}{\bar{p}_3}\left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2}\right)^{\bar{r}_1}\frac{\bar{p}_3{}^{\hat{r}_2}}{\hat{p}_3}\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)^{\hat{r}_1} \geq 1. \tag{17}$$

Recall that $\log : \mathbb{R}_+ \to \mathbb{R}$ is a strictly concave function and use again Walras' law, i. e. $\hat{r}_1 + \hat{r}_2 = 1$. This yields

$$\log\left(\hat{r}_1 \cdot \frac{\bar{p}_3}{\hat{p}_3} + \hat{r}_2\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right)\right) \geq \hat{r}_1 \cdot \log\left(\frac{\bar{p}_3}{\hat{p}_3}\right) + \hat{r}_2 \cdot \log\left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2}\right). \tag{18}$$

Applying the exponential function to equation (18) proves

$$\hat{r}_1 \cdot \frac{\bar{p}_3}{\hat{p}_3} + \hat{r}_2 \left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2} \right) \geq \frac{\bar{p}_3}{\hat{p}_3}^{\hat{r}_1} \left(\frac{\bar{p}_1}{2\,\hat{p}_1} + \frac{\bar{p}_2}{2\,\hat{p}_2} \right)^{\hat{r}_2}. \tag{19}$$

By the same arguments one can show

$$\bar{r}_1 \cdot \frac{\hat{p}_3}{\bar{p}_3} + \bar{r}_2 \left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2} \right) \geq \frac{\hat{p}_3}{\bar{p}_3}^{\bar{r}_1} \left(\frac{\hat{p}_1}{2\,\bar{p}_1} + \frac{\hat{p}_2}{2\,\bar{p}_2} \right)^{\hat{r}_2}. \tag{20}$$

Applying equations (19) and (20) to equation (17) yields $(\hat{p} \cdot \bar{x})(\bar{p} \cdot \hat{x}) \geq 1$, as claimed in (i).

Check that, given normalized incomes $\hat{w} = \bar{w} = 1$, the assumption $\bar{x} \neq \hat{x}$ implies $\bar{q} \neq \hat{q}$. To see this, suppose per absurdum $\bar{q} = \hat{q}$. Recall that for $h_1, h_2 \in \{1, 2, 3\}$ the equation $p_{h_1}/p_{h_2} = q_{h_1}/q_{h_2}$. From the fact $w = p\,(y + \omega)$ and the assumptions $\hat{w} = \bar{w} = 1$ and $\bar{q} = \hat{q}$ and equation (3) on page 6 follows $\hat{r}_i = \bar{r}_i$. Applying the latter result to equation (4) on page 6 yields $\hat{x} = \bar{x}$. This is the desired contradiction. Therefore the assumption $\hat{q} \neq \bar{q}$ is false and the claimed fact $\bar{q} \neq \hat{q}$ is established.

Now it is easy to show that $(\hat{p} \cdot \bar{x})(\bar{p} \cdot \hat{x}) \neq 1$: Suppose, per absurdum $(\hat{p} \cdot \bar{x})(\bar{p} \cdot \hat{x}) = 1$. Then the inequalities (19) and (20) have to be fulfilled with equity. The fact that $\bar{r}_1, \bar{r}_2, \hat{r}_1$, and $\hat{r}_2 \in]0, 1[$ and the strict concavity of the log–function allows to deduce from $(\hat{p} \cdot \bar{x})(\bar{p} \cdot \hat{x}) = 1$ that

$$\frac{\bar{p}_3}{\hat{p}_3} = \frac{\frac{\bar{p}_1}{\hat{p}_1}}{2} + \frac{\frac{\bar{p}_2}{\hat{p}_2}}{2} \quad \text{and} \quad \frac{1}{\frac{\bar{p}_3}{\hat{p}_3}} = \frac{1}{2\frac{\bar{p}_1}{\hat{p}_1}} + \frac{1}{2\frac{\bar{p}_2}{\hat{p}_2}} \tag{21}$$

holds. Equations (21) implies

$$\frac{1}{\frac{\frac{\bar{p}_1}{\hat{p}_1}}{2} + \frac{\frac{\bar{p}_2}{\hat{p}_2}}{2}} = \frac{1}{2\frac{\bar{p}_1}{\hat{p}_1}} + \frac{1}{2\frac{\bar{p}_2}{\hat{p}_2}}. \tag{22}$$

Equation (22) simplifies to

$$\frac{2}{\frac{\bar{p}_1}{\hat{p}_1} + \frac{\bar{p}_2}{\hat{p}_2}} = \frac{\frac{\bar{p}_1}{\hat{p}_1} + \frac{\bar{p}_2}{\hat{p}_2}}{2\,\frac{\bar{p}_1}{\hat{p}_1}\,\frac{\bar{p}_2}{\hat{p}_2}}, \tag{23}$$

that equals

$$4\,\frac{\bar{p}_1}{\hat{p}_1}\,\frac{\bar{p}_2}{\hat{p}_2} = \left(\frac{\bar{p}_1}{\hat{p}_1} + \frac{\bar{p}_2}{\hat{p}_2} \right)^2 \qquad \text{or simply} \qquad \left(\frac{\bar{p}_1}{\hat{p}_1} - \frac{\bar{p}_2}{\hat{p}_2} \right)^2 \tag{24}$$

which yields (ii): $\bar{p}_1/\hat{p}_1 = \bar{p}_2/\hat{p}_2$. Applying equation (ii) to the left equation, displayed at (21), permits to conclude (iii): $\bar{p}_1/\hat{p}_1 = \bar{p}_3/\hat{p}_3$. Using the fact $p_{h_1}/p_{h_2} = q_{h_1}/q_{h_2}$, for $h_1, h_2 \in \{1, 2, 3\}$, one admits readily that p at equations (ii) and (iii) can be replaced by q, that is (iv): $\bar{q}_1/\hat{q}_1 = \bar{q}_2/\hat{q}_2$ and (v): $\bar{q}_1/\hat{q}_1 = \bar{q}_3/\hat{q}_3$. To analyze the consequences of the equations (iv) and (v), consider three cases: (a) $\bar{q}_1 > \hat{q}_1$, (b) $\bar{q}_1 < \hat{q}_1$, and (c) $\bar{q}_1 = \hat{q}_1$. In case (a) equation (iv) implies $\bar{q}_2 > \hat{q}_2$ and equation (v) implies $\bar{q}_3 > \hat{q}_3$. These three inequalities can be summarized as $\bar{q} > \hat{q}$. But $\bar{q} > \hat{q}$ can not happen, since $\hat{q}, \bar{q} \in P$. Thus case (a) may not occur. By the same argument case (b) reveals to be impossible. Now consider case (c). The consequence of the equations (iv) and (v) is $\bar{q} = \hat{q}$. This is the only logical possible case. However, $\bar{q} = \hat{q}$ is the desired contradiction to the proven fact $\bar{q} \neq \hat{q}$. Thus, the initial assumption $(\hat{p} \cdot \bar{x})(\bar{p} \cdot \hat{x}) = 1$ must be false, so that from inequality (i) follows $(\hat{p} \cdot \bar{x})(\bar{p} \cdot \hat{x}) > 1$.

As already mentioned, this is to say that the demand function x^ϵ fulfills the weak axiom.

Q. E. D.

Lemma 3 *Let the sets $\hat{Y}$ and Y^0 be defined by the equations (12) and (13), $y^* \in Y^0$ and $\hat{p} \stackrel{def}{=} (\hat{p}_1, \hat{p}_2, \hat{p}_3) \in \mathcal{N}(\mathcal{T}, y^*)$, then*

1. $\hat{p}_1 = -\hat{p}_2$;

2. cone$\big(\mathrm{conv}\{\hat{p}, p^{lin}\}\big) = \mathcal{N}(\hat{Y}, y^*)$*, the set* $\hat{Y}$ *is regular, verifies assumption 11, and*

3. $\mathcal{N}(\hat{Y}, y^*) \subset \mathcal{N}(Y^0, y^*)$*;*

Proof (1) A glance at equation (10) on page 8 makes clear, that the function g_2 may be viewed as a function $\tilde{g}_2 : \mathbb{R}^2 \to \mathbb{R}$ that depends on only two variables, namely $\tilde{g}(y_1 - y_2, y_3) \stackrel{def}{=} g(y_1, y_2, y_3)$. Thus, for the partial derivatives of g_2 the following holds: $\partial\, g_2/\partial\, y_1 = -\partial\, g_2/\partial\, y_2$. This is equivalent to $\hat{p}_1 = -\hat{p}_2$.

(2) Have a glance at figure 3. The vectors A, B, and C are elements of the set $\partial\mathcal{T}$ and the vectors A, B, and D are contained in Y^{lin}. Thus, by definition of the set Y^0 at equation (13) on page 8, the vectors A and B are elements of the set Y^0. The three vectors $\hat{p}$ are normal to the set $\mathcal{T}$ at the points A, B, and C and the three vectors p^{lin} are normal to the set Y^{lin} at the points A, B, and D. At A and B a convex set with boundaries p^{lin} and $\hat{p}$ is shaded. Casually speaking this is the normal cone to the set $\hat{Y}$ at points A and B. More generally and more precisely: the functions g_1 and g_2, defined at equation (7) and equation (10) on page 7, are differentiable at y^*. From property (1) of this lemma and from the definition of p^{lin} at equation (6) on page 7 follows, that $\hat{p}$ and p^{lin} are positively, linearly independent in the sense of Clarke (1983, p. 55-57), theorem 2.4.7, corollary 2. Thus, Clarke's corollary 2 establishes the claimed facts.

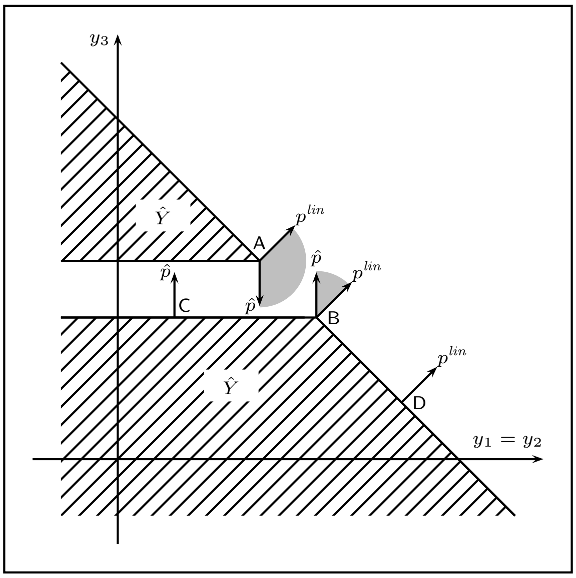

Figure 3: A cut through the production set

(3) From the fact that the functions g_1 and g_2 are smooth follows that p^{lin} and $\hat{p}$ are perpendicular to the sets Y^{lin} and $\mathcal{T}$ at point y^*, in the sense of Clarke (1983, p. 11f.). From the definition of perpandicularity follows, that p^{lin} and $\hat{p}$ are also perpendicular to any subset of both sets Y^{lin} and $\mathcal{T}$ that contains y^*. Thus, from the definition of the set Y^0 follows that the vectors p^{lin} and $\hat{p}$ are perpendicular to the set Y^0 at y^*. From the definition of normality in the sense of Clarke follows p^{lin} and $\hat{p}$ are also normal to the set Y^0 at y^*. The fact that $\mathcal{N}(Y^0, y^*)$ is a convex cone with vertex zero together with property (2) of this lemma implies the claimed inclusion.

Q. E. D.

Lemma 4 *Let* $\epsilon = 0$ *and the consumers be defined as in lemma 1, the set* Y^{lin} *as at equation (6) and the resources* ω *as at equation (5), i. e.* $\mathcal{E}^{lin} \stackrel{def}{=} \big((\mathbb{R}^3_+, \preceq_i, r_i^0), Y^{lin}, \omega\big)$*. Further, let* p^{lin} *and* y^{lin} *be given as at equation (6) on page 7,* $x_1^{lin} \stackrel{def}{=} (0, 0, 2\sqrt{2})$ *and* $x_2^{lin} \stackrel{def}{=} (2, 2, 0)$*. Then* $\big(p^{lin}, (x_1^{lin}, x_2^{lin}), y^{lin}\big)$ *is the unique marginal (cost) pricing equilibrium of the economy* $\mathcal{E}^{lin}$*.*

Proof To see that $\big(p^{lin}, (x_1^{lin}, x_2^{lin}), y^{lin}\big)$ is an equilibrium, first notice that by definition of the set Y^{lin}, at equation (6) on page 7, $y^{lin} \in \partial Y^{lin}$ and $p^{lin} \in \mathcal{N}(Y^{lin}, y^{lin}) \setminus \{0\}$. This is to say that property (ii) of definition 1 on page 5 is verified. From the facts $p^{lin} > 0$ and $p^{lin} \cdot y^{lin} = 8$ follows $w > 0$ and, by lemma 1 on page 6, that the demands x_i^{lin} fulfill property (i) of an equilibrium. By definition at lemma 1 for $\epsilon = 0$ the total demand, denoted by x^{lin}, simplifies to equation (1). Replacing w and p by the values of p^{lin} and w^{lin} yields $x^{lin} = y^{lin}$. Thus, condition (iii) of the definition 1 on page 5 holds and the vector $\big(p^{lin}, (x_1^{lin}, x_2^{lin}), y^{lin}\big)$ is shown to be an equilibrium.

To see the uniqueness simply notice that cone$\big(p^{lin}\big)$ verifies cone$\big(p^{lin}\big) = \mathcal{N}(Y^{lin}, y)$ for all $y \in \partial Y^{lin}$. Therefore the equilibrium price vector is unique (up to a scalar multiplier). Thus, by the definition of the production set Y^{lin}, equilibrium profits are unique (up to the scalar) and individual equilibrium revenues are well defined by equation (3) on page 6. This yields unique individual demands in equilibrium, with the values x_i^{lin} as stated in the lemma. Unique individual demands in equilibrium aggregate to a unique total demand in equilibrium. Since the supply equals the total demand in equilibrium, by condition (iii) of an equilibrium, supply is also uniquely defined.

Q. E. D.

Lemma 5 *Let the functions a and g_2 and the set $\mathcal{T}$ be defined by the equations (9), (10) and (11). Further, let $\hat{\mathcal{E}}^\epsilon$ be the economy defined at theorem 1 on page 8 and $\hat{\mathcal{E}}^0$ be the economy $\hat{\mathcal{E}}^\epsilon$ with $\epsilon = 0$. Then*

1. *the function $a : D_a \to \mathbb{R}$ is well defined and continuous,*
2. *the function $g_2 : \mathbb{R}^3 \to \mathbb{R}$ is well defined and continuous. Further, $g_2(y) > -1/4$ implies (i): $y_3 \in [2.56, 3.24]$ and (ii): $y_1 - y_2 \in [-0.6, 0.6]$.*
3. *the inclusion $Y^0 \subset \mathbb{R}^3_{++}$ holds,*
4. *the set $\hat{Y}$ satisfies free elimination, i. e. assumption 2, and*
5. *if $\big(p, (x_i), y\big)$ is a marginal (cost) pricing equilibrium of the economy $\hat{\mathcal{E}}^0$ or $\hat{\mathcal{E}}^\epsilon$, then $y \in Y^0$,*

Proof (a) By definition of the set D_a the fraction in the square root needed to define a, i. e.

$$\sqrt{\frac{\sqrt{2}\,y_3 - 8 - (y_1 - y_2)}{\underbrace{\sqrt{2}\,y_3}_{\in[3,5]} - 8 + \underbrace{(y_1 - y_2)}_{\in[-1,1]}}}, \tag{25}$$

is restricted as indicated under the braces of equation (25). Thus, the numerator and the denominator of that fraction are both negative, so that the function a is well defined on D. The continuity of the function a, under these restrictions, is obvious. (2) In case the domain of g_2 is restricted to D it follows from the first assertion of this lemma that the function g_2 is well defined and continuous. In case $y \notin D$ these facts are trivial, because the function is constant. It remains to checked that both cases fit together smoothly. Therefore, it has be to shown that $y \in \partial D \Rightarrow g(y) = \frac{1}{4}$. To see this let $v = (a, b)$ be a vector, M be a matrix and $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

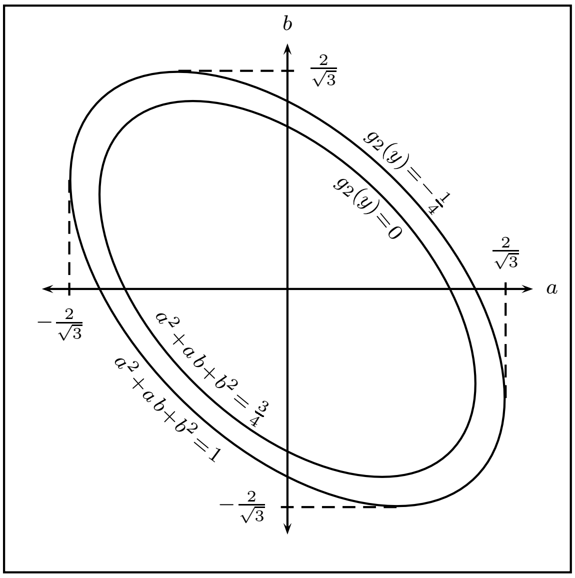

Figure 4: Some ellipses

$$M = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

and $-\tilde{g}_2 = v \cdot (M \cdot v') - 3/4$. The matrix M is symmetric with positive determinant for M and its major sub-matrixes, which implies that the matrix M is positive semi definite. Therefore, the function $-\tilde{g}_2(a, b)$ is convex, so that $\tilde{g}_2(a, b)$ is concave. The function $\tilde{g}_2(a, b)$ expands to $3/4 - a^2 - a\,b - b^2$, which equals g_2. Thus, the assertion $y \in \partial D \Rightarrow g(y) = \frac{1}{4}$ is true if the restriction to D_a is not binding. That is to say that $y \in \{a^2 + a\,b + b^2 \leq 1\}$ should imply $y_1 - y_2 \in [-1, 1]$ and $\sqrt{2}\,y_3 \in [3, 5]$. Here is how to see these inclusions. For all $\lambda > 0$ the implicit function $a^2 + a\,b + b^2 = \lambda$ is an ellipses. For the values $\lambda = 3/4$ and $\lambda = 1$, that is for $g_2 = 0$ and $g_2 = -1/4$, these ellipses are displayed at figure 4. From the fact that the function g_2 is concave in (a, b) follows that $a^2 + a\,b + b^2 \le 1$ implies $a, b \in [-2/\sqrt{3}, 2/\sqrt{3}]$. By definition of b at equation (9), the fact $b \in [-2/\sqrt{3}, 2/\sqrt{3}]$ is equivalent to $y_3 \in [2\,(15\,\sqrt{2} - \sqrt{3})/15, 2\,(15\,\sqrt{2} + \sqrt{3})/15]$. Therefrom it is easy to see that the facts $\sqrt{2} \in [1.4, 1.5]$ and $\sqrt{3} \in [1.7, 1.8]$ imply (i): $y_3 \in [2.56, 3.24]$, as stated in the lemma and $\sqrt{2}\,y_3 \in [3.584, 4.86] \subset [3, 5]$, as displayed under the left brace of equation (25). Solving the two equations $a = \pm 2/\sqrt{3}$ for the variable $y_1 - y_2$ proves that $a \in [-2/\sqrt{3}, 2/\sqrt{3}]$ implies

$$y_1 - y_2 \in \left[\frac{8 + 80\sqrt{3} - \sqrt{2}y_3 - 10\sqrt{6}y_3}{10\sqrt{3} + 151}, \frac{-8 + 80\sqrt{3} + \sqrt{2}y_3 - 10\sqrt{6}y_3}{10\sqrt{3} - 151} \right].$$

Using again the facts $\sqrt{2} \in [1.4, 1.5]$, $\sqrt{3} \in [1.7, 1.8]$ and $y_3 \in [2.56, 3.24]$ easily gives (ii) $y_1 - y_2 \in [-0.6, 0.6]$ as stated in the lemma, and $y_1 - y_2 \in [-1, 1]$, as stated under the right brace of equation (25). This proves $y \in \partial D \Rightarrow g(y) = \frac{1}{4}$.

(3) The second assertion of this lemma establishes that $g_2(y) > -1/4$ implies (i): $y_3 \in [2.56, 3.24]$ and (ii): $y_1 - y_2 \in [-0.6, 0.6]$. By definition at equations (6)–(8) the equation $g_1 = 0$ is equivalent to (iii): $y_1 + y_2 + \sqrt{2}\,y_3 = 8$. Subtracting y_3, as restricted by inequality (i), from (iii) and applying the fact $\sqrt{2} \in [1.4, 1.5]$ permits to conclude (iv): $y_1 + y_2 \in [2.87, 4.416]$. Inequality (ii) implies $y_2 \in [y_1 - 0.6, y_1 + 0.6]$. Therefore (v): $y_1 \in [1.135, 2.508]$. Applying again (ii) yields (vi): $y_2 \in [0.535, 3.108]$. The inequalities (i), (v), and (vi) prove that $g_1(y) = g_2(y) = 0$ implies $y > 0$.

(4) The second assertion of this lemma permits to conclude $(y_1, y_2, y_3) \in \mathbb{R}^3 \setminus \mathcal{T} \Rightarrow y_3 > 0$. Therefore, $-\mathbb{R}^3_+ \subset \mathcal{T}$. By definition of the set Y^{lin} at equation (8) on page 7 the inclusion $-\mathbb{R}^3_+ \subset Y^{lin}$ holds. Thus the set $-\mathbb{R}^3_+$ is contained in the intersection $\mathcal{T} \cap Y^{lin}$. This intersection defines the set $\hat{Y}$, see equation (12) on page 8.

(5) Notice that $y \in \partial\hat{Y}$ implies one of the following three cases: (i): $g_1 < 0$ and $g_2 = 0$, (ii): $g_1 = 0$ and $g_2 < 0$, or (iii): $g_1 = 0$ and $g_2 = 0$. In that follows it is shown that no equilibrium occurs in both first cases. (i) In case $g_1 < 0$ and $g_2 = 0 : y \in \partial\mathcal{T}$. Therefore, the first assertion of lemma 3 guarantees that for the equilibrium price (p_1, p_2, p_3) verifies $p_1 = -p_2$. However, from assumption 8 on page 5 (boundary behavior), verified to hold at assertion 1 of lemma 2 on page 9, follows that the equilibrium price is strictly positive, i. e. $p > 0$. This is the desired contradiction to $p_1 = -p_2$. Therefore, in this case no equilibrium may occur. (ii) In case $g_1 = 0$ and $g_2 < 0$ an equilibrium $\big(p, (x_i), y\big)$ would also be an equilibrium of the linear economy $\mathcal{E}^{lin}$. From lemma 4 on page 11 follows that $y = y^{lin}$. From simple calculus follows $g_2(y^{lin}) = 3/4$, which is strictly positive. But this is a contradiction to the assumption $g_2(y) < 0$. Thus even in this case no equilibrium may occur. This permits to conclude that equilibria may occur only in case (iii), that is for $g_1 = 0$ and $g_2 = 0$. Thus, in equilibrium $y \in Y^0$, as claimed in the final assertion of this lemma. Q. E. D.

Lemma 6 *Let ω be the vector and $\hat{Y}$ be the set defined at the equations (5) and (12). Let for all $\bar{\omega} \ge \omega$ the attainable set $\hat{\mathcal{A}}(\bar{\omega})$ be defined by $\{y \in \hat{Y} \mid y + \bar{\omega} \ge 0\}$. Then the attainable set $\hat{\mathcal{A}}(\bar{\omega})$ is bounded, i. e. assumption 5 is verified.*

Proof The fact, that the attainable set is bounded for all $\bar{\omega} \ge \omega$, is well known for linear production sets such as Y^{lin}. Therefore, the attainable set is bounded for all subsets, especially for $\hat{Y} \subset Y^{lin}$. Q. E. D.

Lemma 7 *Let Y^{lin}, $\hat{Y}$, and Y^0 be the sets defined by equation (8), (12), and (13). Suppose $(p, y) \in \mathbb{R}^3_+ \times \partial^+\hat{Y}$ and $p \in \mathcal{N}(\hat{Y}, y) \setminus \{0\}$. Then $p \cdot y > 0$, i. e. assumption 10 is verified.*

Proof To prove, that the economy $\hat{\mathcal{E}}^{\epsilon}$ satisfies the survival assumption, it is useful to consider the following cases: (a): $g_1 = 0$ and $g_2 = 0$, (b): $g_1 = 0$ and $g_2 < 0$, and (c): $g_1 < 0$ and $g_2 = 0$. (a) From property 2 of lemma 5 on the preceding page it is known that $\forall y \in Y^0 : y > 0$. Thus, for all $p \in \mathbb{R}^3_+ \setminus \{0\}$ the inequality $p \cdot y > 0$ holds, as stated. (b) In case $g_1 = 0$ and $g_2 < 0 : y \in \partial Y^{lin}$. The assumption $p \in \mathcal{N}(\hat{Y}, y) \setminus \{0\}$ implies $t \cdot p = p^{lin}$, with p^{lin} defined at equation (6) on page 7 and $t \in \mathbb{R}_{++}$. Thus $p \cdot y = t \cdot 8$, which is strictly positive, as claimed. (c) Assertion (1) of lemma 3 on page 10 guarantees that in this case (i): $p_1 = -p_2$. The assumption $(p_1, p_2, p_3) \geq 0$ implies $p_1 = p_2 = 0$. Thus $p \geq 0$ and $p \neq 0$ implies (ii): $p_3 > 0$. Finally, assertion 1 of lemma 5 on page 12 insures (iii): $y_3 > 0$. Notice that (i), (ii), and (iii) imply $p \cdot y > 0$. Q. E. D.

2.3 An equivalent economy

It is useful to define the set Y^0 parametrically. For logical reasons the parametrically defined set is named Y, and the equity $Y = Y^0$ is proven.

In that follows some trigonometric functions are used. To shorten notation recall that the interval $[-\pi, \pi]$ is denoted by I and the trigonometric functions sin, cos : $I \to \mathbb{R}$ are abbreviated by s and c.

Lemma 8 *Let the functions ψ, y_3, y_1, and $y_2 : I \to \mathbb{R}$ be defined by*

$$\psi \overset{def}{=} 1 + \frac{\cos\alpha}{10}, \qquad y_3 \overset{def}{=} 2\sqrt{2} - \frac{\sin(\alpha - \pi/6)}{5}, \tag{26}$$

$$y_1 \overset{def}{=} (8 - \sqrt{2}\, y_3)\frac{\psi^2}{1+\psi^2}, \qquad y_2 \overset{def}{=} (8 - \sqrt{2}\, y_3)\frac{1}{1+\psi^2}. \tag{27}$$

Further, let Y^0 and Y be the sets defined at equation (13) on page 8 and $Y \overset{def}{=} \{(y_h(\alpha)) \in \mathbb{R}^3 | \alpha \in [-\pi, \pi]\}$. Then (1): $\sqrt{y_1/y_2} = \psi$ and (2): $Y = Y^0$.

See figure 5 for a geometric intuition of the set Y^0. Notice, that only the components y_1 and y_2 were computed, i. e. it is a projection of Y^0 onto the plan $\{(y_1, y_2) | (y_1, y_2) \in \mathbb{R}^2\}$.

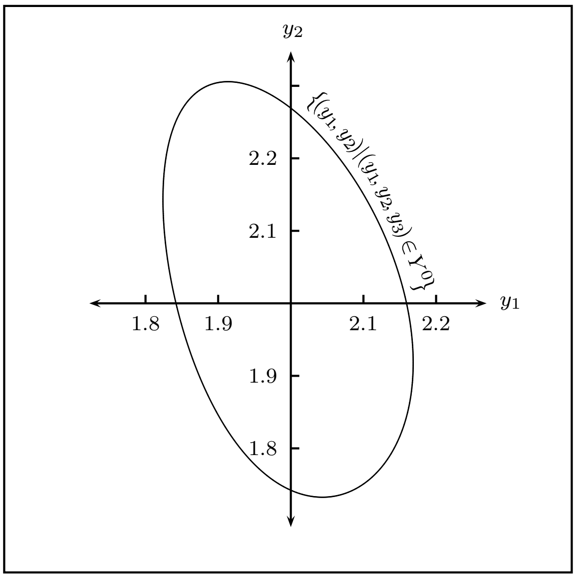

Figure 5: A projection of the technology

Proof (1) The fact $\sqrt{y_1/y_2} = \psi$ follows readily from the definition of y_1 and y_2 at equation (27). It is stated here for later reference.

(2) It has to be shown that (a): $Y^0 \subset Y$ and (b): $Y \subset Y^0$. To see (a) let $\tilde{y} : g_1(\tilde{y}) = g_2(\tilde{y}) = 0$. Recall that $g_1(\tilde{y}) = 0$ implies $\tilde{y}_1 + \tilde{y}_2 + \sqrt{2}\,\tilde{y}_3 = 8$. In this case the value of a reduces to a_1, a_1 defined by $10\,(\sqrt{y_1/y_2} - 1)$. Thus $g_2(\tilde{y}) = 0$ implies $a_1^2(\tilde{y}) + a_1(\tilde{y}) \cdot b(\tilde{y}) + b^2(\tilde{y}) = 3/4$. This is a point of the ellipsis for $a_1(\alpha) = \cos\alpha$ and $b(\alpha) = \sin(\alpha - \pi/6)$, as can be seen applying the trigonometric addition theorems and the facts $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$ yields (ii): $b = (\sqrt{3}\,s - c)/2$. From (i) and (ii) follows $a_1^2 = c^2$, $b^2 = (\sqrt{3}\,s - c)^2/4$ and $a_1 \cdot b = c\,(\sqrt{3}\,s - c)/2$. Thus $a_1^2 + a_1 \cdot b + b^2 = 3\,(c^2 + s^2)/4 = 3/4$. Thus there exists an $\alpha \in [-\pi, \pi]$ such that $a_1(\alpha) = a_1(\tilde{y})$ and $b(\alpha)) = b(\tilde{y})$. This proves $Y^0 \subset Y$. (b) To see $Y \subset Y^0$, fix $\tilde{\alpha} \in [-\pi, \pi]$. From equation (27) follows $y_1(\tilde{\alpha}) + y_2(\tilde{\alpha}) = 8 - \sqrt{2}\,y_3(\tilde{\alpha})$. Thus $y_1(\tilde{\alpha}) + y_2(\tilde{\alpha}) = \sqrt{2}\,y_3(\tilde{\alpha}) = 8$, which is equivalent to (iii): $g_1(y(\tilde{\alpha})) = 0$, by equation (7). It has already been seen in (a) that under this assumption for $a_1(\tilde{\alpha}) = \cos\tilde{\alpha}$ and $b(\tilde{\alpha}) = \sin(\tilde{\alpha} - \pi/6)$ the equation $a_1^2(y(\tilde{\alpha})) + a_1(y(\tilde{\alpha})) \cdot b(y(\tilde{\alpha})) + b^2(y(\tilde{\alpha})) = 3/4$

holds. The latter equation is equivalent to (iv): $g_2\big(y(\tilde{\alpha})\big) = 0$ by equation (10). Equations (iii) and (iv) imply $y(\tilde{\alpha}) \in Y^0$, or $Y \subset Y^0$

Q. E. D.

Lemma 9 *Let $\hat{\mathcal{E}}^\epsilon$ be the economy defined at theorem 1 on page 8, and $\mathcal{E}^0$ be the economy defined by $\mathcal{E}^0 \stackrel{def}{=} \big((\mathbb{R}^3_+, \preceq_i, r_i^0), Y, \omega\big)$. If the economy $\mathcal{E}^0$ does not admit an equilibrium, then there exists an $\epsilon > 0$ such that the economy $\hat{\mathcal{E}}^\epsilon$ also does not admit an equilibrium.*

Proof This lemma is proven by contra position. That is to say, that there exists a sequence $\{\epsilon^\nu\}_{\nu=1}^\infty$ converging to zero, such that the economies $\hat{\mathcal{E}}^{\epsilon^\nu}$ admit equilibria. The lemma is proven, if it can be deduced from this hypothesis, that the economy $\mathcal{E}^0$ also admits an equilibrium.

From the hypothesis that equilibria of the economies $\hat{\mathcal{E}}^{\epsilon^\nu}$ exist follows, that a sequence of equilibria exist, denoted by $\{(p^\nu, (x_i^\nu), y^\nu)\}_{\nu=1}^\infty$. The proof is finished if it is shown that the sequence $\big\{\big(p^\nu, (x_i^\nu), y^\nu\big)\big\}_{\nu=1}^\infty$ converges towards some vector $\big(\bar{p}, (\bar{x}_i), \bar{y}\big)$ and that $\big(\bar{p}, (\bar{x}_i), \bar{y}\big)$ is an equilibrium of the economy $\mathcal{E}^0$. This proof takes three steps. (1) First it will be seen that $(\bar{p}, \bar{y})$ exists and verifies $\bar{y} \in \partial Y$ and $\bar{p} \in \mathcal{N}(Y, \bar{y}) \setminus \{0\}$, i. e. condition (ii) of definition 1 on page 5. (2) Second, the existence of incomes $\bar{r}_i$, defined by $\bar{r}_i \stackrel{def}{=} r_i^\epsilon\big(\bar{\epsilon}, \bar{p}, (\bar{p} \cdot \bar{y})\big)$ and the existence of the individual demands $\bar{x}_i$, defined by $\bar{x}_i = x_i(\bar{r}_i, \bar{p})$ is established and it is shown that $\bar{x}_i$ is a greater element for $\preceq_i$ in the budget set $\{x \in X_i | \bar{p} \cdot x \leq \bar{r}_i\}$, i. e. $\bar{x}_i$ verifies condition (i) of definition 1 on page 5 for the price $\bar{p}$ at income $\bar{r}_i$. (3) Finally it is proven that markets are clear, i. e. $(\bar{x}, \bar{y})$ verifies condition (iii) of an equilibrium.

(1) At lemma 5 on page 12, property 5 it is proven that $y^\nu \in Y^0$. Let $\{\alpha^\nu\}_{\nu=1}^\infty$ be the sequence, defined by $\alpha^\nu : y(\alpha^\nu) = y^\nu$. Since $\alpha^\nu \in I$, there exists a converging subsequence with limit $\bar{\alpha}$. In that follows only this subsequence is considered. As a consequence of the continuity of the function $y(\alpha)$, as stated at equation (26) and (27), the sequence of productions y^ν converges to $\bar{y} = y(\bar{\alpha})$. Property 2 of lemma 3 on page 10 establishes that the set $\hat{Y}$ is regular. From the assumption that $\big(p^\nu, (x_i^\nu), y^\nu\big)$ are equilibria of the economies $\hat{\mathcal{E}}^\epsilon$ follows that condition (ii) of definition 1 on page 5, that is $p^\nu \in \mathcal{N}(Y, y^\nu) \setminus \{0\}$. It follows from Clarke (1983, p. 58, corollary), that a vector $\bar{p}$, defined by $\bar{p} \stackrel{def}{=} \lim_{\nu\to\infty} p^\nu$ exists and $\bar{p} \in \mathcal{N}(\hat{Y}, \bar{y}) \setminus \{0\}$. From property 3 of lemma 3 on page 10 follows $\bar{p} \in \mathcal{N}(Y, \bar{y}) \setminus \{0\}$. Thus there exist $(\bar{p}, \bar{y})$ that fulfill property (ii) of an equilibrium, i. e. definition 1 on page 5, for the economies $\hat{\mathcal{E}}^0$ and $\mathcal{E}^0$.

(2) Let the total incomes w^ν and $\bar{w}$ be defined by $w^\nu \stackrel{def}{=} p^\nu \cdot y^\nu$, $\bar{w} \stackrel{def}{=} \bar{p} \cdot \bar{y}$. From the assumption that the vectors $\big(p^\nu, (x_i^\nu), y^\nu\big)$ are equilibria and from the fact that the demand satisfies boundary behavior follows $p^\nu > 0$. From the fact $\bar{y} \in Y^0$ follows, by assertion 2 of lemma 5 on page 12, $\bar{y} > 0$. Thus $\bar{w}$ is strictly positive. Therefore, $\bar{p} \geq 0$ and $\bar{p} \neq 0$ holds. From the fact that p^ν are equilibrium prices and because the demand satisfies boundary behavior follows $\bar{p} > 0$. The parameterized revenue functions r_i^ϵ are continuous for strictly positive prices p^ν, $\bar{p}$ and total incomes w^ν and $\bar{w}$ and all $\epsilon \in [0, 1/\sqrt{3}]$. Therefrom follows that the limits $\bar{r}_i = \lim_{\nu\to\infty} r_i^\nu$ exist and incomes r_i^ϵ are strictly positive for strictly positive total income $\bar{w}$ and strictly positive prices $\bar{p}$, so that $r_i^\nu > 0$ and $\bar{r}_i > 0$ holds. The parameterized individual demand functions x_i^ϵ are continuous for strictly positive prices $\bar{p}$ and p^ν and strictly positive individual incomes $\bar{r}_i$ and r_i^ν, for all $\epsilon \in [0, 1/\sqrt{3}]$. Therefore, the limits $\bar{x}_i \stackrel{def}{=} \lim_{\nu\to\infty} x_i^\epsilon(\epsilon^\nu, p^\nu, r_i^\nu)$ exist and verify $\bar{x}_i = x_i^\epsilon(\bar{\epsilon}, \bar{p}, \bar{r}_i)$. Lemma 1 on page 6 guarantees that individual demands verify property (i) of an equilibrium, i. e. definition 1 on page 5, for the economies $\mathcal{E}^0$ and $\hat{\mathcal{E}}^0$.

(3) Finally, notice that markets clear, i. e. $\bar{x}_1 + \bar{x}_2 = \bar{y} + \omega$. To see this suppose, again per absurdum, the contrary. In that case, again due to continuity of the supply and the demand, markets would have been uncleared for sufficiently large ν for $\big((x_i^\nu), y^\nu\big)$. But this is the desired contradiction to the assumption that $\big(p^\nu, (x_i^\nu), y^\nu\big)$ are equilibria for all ν. Thus, for $\big((\bar{x}_i), \bar{y}\big)$ the market clearing condition of an equilibrium, i. e. definition 1 on page 5, for the economies $\mathcal{E}^0$ and $\hat{\mathcal{E}}^0$ is verified.

Thus, from the absurd assumption that equilibria exist for all ϵ^ν one can deduce that the limit of the sequence of these equilibria $\big(\bar{p}, (\bar{x}_i), \bar{y}\big)$ would be an equilibrium of the economies $\hat{\mathcal{E}}^0$ and $\mathcal{E}^0$. This proves the lemma.

Q. E. D.

2.4 The non–existence result

Lemma 10 *The economy $\mathcal{E}^0$ does not admit an equilibrium.*

Proof One of the nice properties of Shafer's demand function is that from equation (1) follows immediately $p_1\, x_1 = p_2\, x_2$, $p_1/p_2 = x_2/x_1$. This permits to derive the inverse demand denoted by $\bar{\wp}$ with $\bar{\wp} : \mathbb{R}^4_{++} \to \mathbb{R}^3_{++}$, to equal

$$\begin{aligned} \bar{\wp}_i &= \frac{w}{2\, x_i\left(1+\sqrt{\frac{x_1}{x_2}}\right)}, \quad i = 1,2 \\ \bar{\wp}_3 &= \frac{w}{x_3\left(1+\sqrt{\frac{x_2}{x_1}}\right)}. \end{aligned} \tag{28}$$

Using the homogeneity of the demand function, i. e. assumption 7 on page 5, prices can be normalized such that $p \cdot x = 1 + \sqrt{x_1/x_2}$. This allows to define the *normalized* demand and inverse demand function ξ, $\wp : \mathbb{R}^3_{++} \to \mathbb{R}^3_{++}$, to equal

$$\xi = \left(\frac{1}{2\,p_1}, \frac{1}{2\,p_2}, \frac{1}{p_3}\cdot\sqrt{\frac{p_2}{p_1}}\right) \quad \text{and} \quad \wp = \left(\frac{1}{2\,x_1}, \frac{1}{2\,x_2}, \frac{1}{x_3}\cdot\sqrt{\frac{x_1}{x_2}}\right). \tag{29}$$

In equilibrium $x = y$, so that the normalized inverse demand, given at equation (29), equals

$$\wp \overset{def}{=} \left(\frac{1}{2\,y_1}, \frac{1}{2\,y_2}, \frac{1}{y_3}\cdot\sqrt{\frac{y_1}{y_2}}\right). \tag{30}$$

In equilibrium, the normalized inverse demand function points into the normal cone of the production set at equilibrium production, i. e. $\wp \in \mathcal{N}(Y^0, y)$.

Applying the polarity of Clarke's normal cones and his tangent cones, this is to say that the (normalized) inverse demand in equilibrium is orthogonal to Clarke's tangent cone, i. e. $\wp \cdot y' = 0$, where $y' \overset{def}{=} (y'_h)$ denotes the derivative of the functions y_h, defined at lemma 8 on page 14, with respect to α. That is to say in equilibrium the function $f : I \to \mathbb{R}$ defined by $\wp \cdot y'$ vanishes. Applying the formalization of the normalized inverse demand given at equation (30) the equilibrium condition f equals

$$f = \frac{y'_1}{2\,y_1} + \frac{y'_2}{2\,y_2} + \frac{y'_3}{y_3}\sqrt{\frac{y_1}{y_2}}. \tag{31}$$

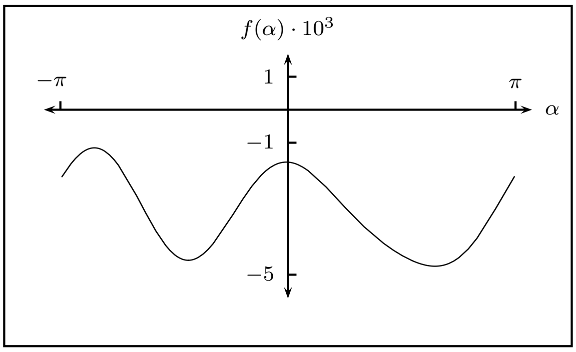

Figure 6: The equilibrium condition

The graph of the equilibrium condition f is displayed at figure 6. If the reader is willing to see that the graph of f is strictly negative, then she/he is convinced that no equilibria exist for the economy $\mathcal{E}^0$.

The remaining of that paper is devoted to a rigorous proof of the fact $f(\alpha) < 0$, for all $\alpha \in I$. From the definition of y_1 and y_2 at equation (27) on page 14 follows

$$\frac{y_1'}{2\,y_1} + \frac{y_2'}{2\,y_2} = \frac{\psi'}{1+\psi^2}\left(\frac{1}{\psi} - \psi\right) + \frac{y_3'}{y_3 - 4\sqrt{2}}, \tag{32}$$

with $\psi' \stackrel{def}{=} d\,\psi/d\,\alpha$. Further, by property 1 of lemma 8 on page 14, the $\sqrt{y_1/y_2}$ in the function f, as defined at equation (31), can be replaced by ψ. Applying equation (32) this is to say

$$f = \underbrace{\frac{\psi'}{1+\psi^2}\left(\frac{1}{\psi} - \psi\right)}_{f_\ell} + \underbrace{\frac{y_3'}{y_3 - 4\sqrt{2}} + \frac{\psi\,y_3'}{y_3}}_{-f_u}. \tag{33}$$

If the functions f_u and $f_\ell : I \to \mathbb{R}$ are defined as indicated under the braces of equation (33), i. e. $f_u \stackrel{def}{=} y_3'/(4\sqrt{2} - y_3) - \psi \cdot y_3'/y_3$ and $f_\ell \stackrel{def}{=} \psi'\,(1/\psi - \psi)/(1+\psi^2)$, then the lemma is proven if $f_\ell < f_u$. To simplify the proof define the simple function $\varphi : [-\pi, \pi] \to \mathbb{R}$ by

$$\varphi = \frac{1}{1000} + \frac{s \cdot c}{100}, \tag{34}$$

that will be shown to verify the following two inequalities

$$f_\ell - \varphi < 0 \tag{35}$$

and

$$\varphi - f_u < 0. \tag{36}$$

Of course, the lemma is also proven if inequalities (35) and (36) are shown to hold.

Inequality (35) can be proven as follows: The function ψ is defined at equation (26) on page 14. From that definition follows $\psi' = -s/10$. Replacing ψ and ψ' in the definition of f_ℓ, as given over the left brace at equation (33) on page 17 yields

$$f_\ell = -\frac{s\left(-1 + \frac{1}{1+c/10} - c/10\right)}{10\ (1 + (1+c/10)^2)}.$$

Multiplying the numerator and the denominator by $10\,(10+c)$ expands to

$$f_\ell = -\frac{-100\,s + \frac{100\,s}{1+\frac{c}{10}} - 20\,c \cdot s + \frac{10\,c \cdot s}{1+\frac{c}{10}} - s \cdot c^2}{10\,(10+c)\,(20 + 2\,c + c^2/10)},$$

which simplifies to

$$f_\ell = \frac{c \cdot s\,(20+c)}{(10+c)\,\big(200 + c\,(20+c)\big)}. \tag{37}$$

Next, recall that $\sqrt{3} \approx 1.73$, to see that inequalities $0 < 1 < 5/3 < \sqrt{3}$ hold. These imply $0 < 1\big/\sqrt{3} < 6/10$ and $0 < 2\big/\big(3\sqrt{3}\big) < 4/10$. Thus

$$\frac{2}{3}\sqrt{1 - \frac{2}{3}} < \frac{4}{10}. \tag{38}$$

Notice that $d(s \cdot c^2)/d\,\alpha = c^3 - 2\,c \cdot s^2$ or $c\,(3\,c^2 - 2)$. The latter form allows to solve the cosines of the roots of $d(s \cdot c^2)/d\,\alpha$, equal to $\tilde{c} = 0$ and $\tilde{c} = \pm\sqrt{2/3}$. Thus, the function $s \cdot c^2$ attains its extrema at some of these three values of $\tilde{c}$. At $\tilde{c} = 0$, trivially $-\tilde{s} \cdot \tilde{c}^2 - 4/10 < 0$. For both other possible extreme values the equality $\tilde{c}^2 = 2/3$ holds. Using $s = \pm\sqrt{1-c^2}$ yields $\tilde{s} \in \{\pm\sqrt{1-2/3}\}$. Thus $\tilde{s} \cdot \tilde{c}^2 = \pm 2\sqrt{1-2/3}/3$. The latter equation, together with inequality (38) implies

$$-s \cdot c^2 - 4/10 < 0. \tag{39}$$

Multiplying inequality (39) by 3000 yields (i): $-1200 - 3000\, s \cdot c^2 < 0$. From the facts $c \in [-1,1]$ and $s \in [-1,1]$ follows readily (ii): $-800 - 400\, c - 30\, c^2 - c^3 - 300\, s \cdot c^3 - 10\, s \cdot c^4 < 0$. Adding inequality (i) and (ii) and dividing by 1000 yields $-2 - 2\, c/5 - 3\, c^2/100 - c^3/1000 - 3\, s \cdot c^2 - 3\, s \cdot c^3/10 - s \cdot c^4/100 < 0$. The terms under the braces of equation (40) can be gained expanding the under-braced expressions. Note that they sum up to the left hand side of last inequality. Thus

$$\underbrace{c \cdot s\,(20+c)}_{20\, c \cdot s + s \cdot c^2} - \underbrace{\frac{(10+c)\big(200+c\,(20+c)\big)}{1000}}_{-2-\frac{c}{5}-\frac{c^2}{100}-\frac{c}{5}-\frac{2\,c^2}{100}-\frac{c^3}{1000}} - \underbrace{\frac{c \cdot s\,(10+c)\big(200+c\,(20+c)\big)}{100}}_{-20\, c \cdot s - 2\, s \cdot c^2 - \frac{s \cdot c^3}{10} - 2\, s \cdot c^2 - \frac{2\, s \cdot c^3}{10} - \frac{s \cdot c^4}{100}} < 0. \tag{40}$$

Recalling that $c,\, s \in [-1,1]$, one admits readily $(10+c)\big(200 + c \cdot (20+c)\big) > 0$. Therefore, dividing inequality (40) by $(10+c)\big(200 + c \cdot (20+c)\big)$ yields

$$\underbrace{\frac{c \cdot s\,(20+c)}{(10+c)\big(200+c\,(20+c)\big)}}_{f_\ell} - \underbrace{\left(\frac{1}{1000} + \frac{c \cdot s}{100}\right)}_{\varphi} < 0.$$

A glance at equation (34) on page 17 and equation (37) on page 17 makes sure that the terms under the braces equal the under-braced parts of the equation. Thus inequality (35) holds.

The proof of inequality (36) is divided into two parts. First the inequalities $\phi_1 < 0$, $\phi_2 < 0$, $\phi_3 < 0$ and $\phi_4 < 0$ are proven, where the functions ϕ_1, ϕ_2, ϕ_3 and $\phi_4 : I \to \mathbb{R}$ are defined by

$$\begin{aligned} \phi_1 &= -44 - 122\sqrt{3} - c^2 + 100\sqrt{3}\, c^3 + 2\sqrt{3}\, c \cdot s - \\ &\quad - 10\, c^3 \cdot s - 3\, s^2 + 20\sqrt{3}\, c^2 \cdot s^2 - 30\, s^3 \cdot c, \\ \phi_2 &= -80 - 200\, c^2 \cdot s, \\ \phi_3 &= -40\sqrt{3} - 100\sqrt{3}\, s^2 \cdot c, \\ \phi_4 &= 924 + 162\sqrt{3} + 2000\sqrt{3}\, c^2 - 2000\sqrt{6}\, c^2 + \\ &\quad + 4000\, c \cdot s - 2000\sqrt{2}\, c \cdot s - 2000\sqrt{3}\, s^2. \end{aligned} \tag{41}$$

In the second part of the proof it is shown that the sum of these inequalities implies inequality (36). Notice that the fact $c, s \in [-1,1]$ implies $\phi_1 < 0$. A glance at inequality (39) on page 18 makes clear that $\phi_2 < 0$. Further, inequality (39) also implies $-c \cdot s^2 - 4/10 < 0$, because of the well known facts $c = \sin(\pi/2 - \alpha)$ and $s = \cos(\pi/2 - \alpha)$. The inequality $-c \cdot s^2 - 4/10 < 0$ follows by substitution of α with $\pi/2 - \alpha$ at inequality (39). One admits readily that $-c \cdot s^2 - 4/10 < 0$ is equivalent to $\phi_3 < 0$. Finally, from the well known approximations $\sqrt{2} > 1.414$ and $\sqrt{3} \in]1.73, 1.74[$ and the fact $924 + 162 \cdot 1.74 - 2000\,(25 \cdot 1.414 + 13 \cdot 1.73 - 50)/13 = -89/325$ follows $924 + 162\sqrt{3} < 2000\,(24\sqrt{2} + 13\sqrt{3})/13$ and $\xi_1 < \xi_2$, with $\xi_1, \xi_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} \xi_1(t) &\overset{def}{=} (924 + 162\sqrt{3})\,(1+t^2) + \\ &\quad + 2000\sqrt{3} - 2000\sqrt{6} + 4000\, t - 2000\sqrt{2}\, t - 2000\sqrt{3}\, t^2 \quad \text{and} \\ \xi_2(t) &\overset{def}{=} \big(2000\,(24\sqrt{2} + 13\sqrt{3})/13\big)\,(1+t^2) + \\ &\quad + 2000\sqrt{3} - 2000\sqrt{6} + 4000\, t - 2000\sqrt{2}\, t - 2000\sqrt{3}\, t^2, \end{aligned}$$

for all $t \in \mathbb{R}$. The first and second order derivatives of the function ξ_2 simplify to $\xi_2'(t) = 2000\,(\sqrt{2} - 2)(50\, t - 13)/13$ and $\xi_2''(t) = 100000\,(\sqrt{2} - 2)/13$. Notice that $\xi_2''(t) < 0$ for all $t \in \mathbb{R}$ and $\xi_2'(13/50) =$

0. That is to say that the function $\xi_2(t)$ attains its unique maximum at $t = 13/50$. Simple calculus shows $\xi_2(13/50) = -20\,(\sqrt{2}-2)\,(1300\sqrt{3}-2331)/13$. From the facts $1.74 > \sqrt{3}$ and $1300 \cdot 1.74 - 2331 = -69$ follows $\xi_2(13/50) < 0$. Thus, for all $t \in \mathbb{R} : \xi_2(t) < 0$. From the inequality $\xi_1 < \xi_2$ follows for all $t \in \mathbb{R}$: $\xi_1(t) < 0$. Let the function $t : I \to \mathbb{R}$ be defined by $t = \tan\alpha$ and define the function $\phi_4^* : I \to \mathbb{R}$ by $\phi_4^* \stackrel{def}{=} \xi_1/(1+t^2)$, then the fact $\xi_1 < 0$ implies $\phi_4^*(\alpha) < 0$ for all $\alpha \in I$. Recall the well known facts $s^2 = t^2/(1+t^2)$, $c^2 = 1/(1+t^2)$, and $c \cdot s = t/(1+t^2)$ to see $\phi_4^* = 924 + 162\sqrt{3} + 2000\sqrt{3}\,c^2 - 2000\sqrt{6}\,c^2 + 4000\,c \cdot s - 2000\sqrt{2}\,c \cdot s - 2000\sqrt{3}\,s^2 < 0$. Compare the latter inequality with the definition of ϕ_4 at equation (41) to see $\phi_4^* = \phi_4$. Thus $\phi_4 < 0$. Therefore, $\sum_{\nu=1}^{4} \phi_\nu$ is strictly negative. This is to say

$$\begin{aligned}800 - c^2 + 2000\sqrt{3}\,c^2 - 2000\sqrt{6}\,c^2 + 100\sqrt{3}\,c^3 + \\ + 4000\,c \cdot s - 2000\sqrt{2}\,c \cdot s + 2\sqrt{3}\,c \cdot s - 200\,c^2 \cdot s - 10\,c^3 \cdot s - 3\,s^2 - \\ - 2000\sqrt{3}\,s^2 - 100\sqrt{3}\,c \cdot s^2 + 20\sqrt{3}\,c^2 \cdot s^2 - 30\,c \cdot s^3 < 0.\end{aligned} \tag{42}$$

Expand inequality (43) to see that it is equivalent to inequality (42):

$$\begin{aligned}\frac{(1+10\,c \cdot s)\,(20\sqrt{2}+c-\sqrt{3}\,s)\,(20\sqrt{2}-c+\sqrt{3}\,s)}{1000} + \\ + (\sqrt{3}\,c+s)\,(20\sqrt{2}+c-\sqrt{3}\,s) - \frac{(10+c)\,(\sqrt{3}\,c+s)\,(20\sqrt{2}-c+\sqrt{3}\,s)}{10} < 0.\end{aligned} \tag{43}$$

One admits readily that $c, s \in [-1, 1]$ implies $(20\sqrt{2}+c-\sqrt{3}\,s)\,(20\sqrt{2}-c+\sqrt{3}\,s) > 0$. Thus inequality (43) is equivalent to

$$\underbrace{\frac{1+10\,c\,s}{1000}}_{\varphi} + \underbrace{\overbrace{\frac{\sqrt{3}\,c+s}{20\sqrt{2}-c+\sqrt{3}\,s}}^{\frac{y_3'}{y_3-4\sqrt{2}}} \overbrace{-\frac{(10+c)\,(\sqrt{3}\,c+s)}{10\,(20\sqrt{2}+c-\sqrt{3}\,s)}}^{-\frac{\psi\,y_3'}{y_3}}}_{-f_u} < 0. \tag{44}$$

Check the definition of y_3 at equation (26) on page 14 and notice that the trigonometric addition theorems imply $10\,y_3 = 20\sqrt{2}+c-\sqrt{3}\,s$. To see this, it is helpful to remember $\sin(-\pi/6) = -1/2$ and $\cos(-\pi/6) = \sqrt{3}/2$. From the latter definition of y_3 one can compute $10\,y_3' = -\sqrt{3}\,c - s$. Thus, the terms over the braces equal the over-braced parts of equation (44). A glance at the definition of f_u and φ at equation (33) and (34) proves that the terms under the braces equal the under-braced parts of equation (44). This proves inequality (36).

Q. E. D.

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